INTERNATIONAL BACCALAUREATE
BACCALAURÉAT INTERNATIONAL
BACHILLERATO INTERNACIONAL

MARKSCHEME

November 1999

MATHEMATICS

Higher Level

Paper 2

23 pages

Paper 2 Markscheme

Instructions to Examiners

1 All marking must be done using a **red** pen.

2 **Abbreviations**

The markscheme may make use of the following abbreviations:

M Marks awarded for **Method**

A Marks awarded for an **Answer** or for **Accuracy**

C Marks awarded for **Correct** statements

R Marks awarded for clear **Reasoning**

AG **Answer Given** in the question and consequently marks are **not** awarded

3 **Follow Through (ft) Marks**

Questions in this paper were constructed to enable a candidate to:

- show, step by step, what he or she knows and is able to do;
- use an answer obtained in one part of a question to obtain answers in the later parts of a question.

Thus errors made at any step of the solution can affect all working that follows. Furthermore, errors made early in the solution can affect more steps or parts of the solution than similar errors made later.

To limit the severity of the penalty for errors made at any step of a solution, **follow through (ft)** marks should be awarded. The procedures for awarding these marks require that all examiners:

(i) penalise an error when it **first occurs**;

(ii) **accept the incorrect answer** as the appropriate value or quantity to be used in all subsequent parts of the question;

(iii) award **M** marks for a correct method, and **A(ft)** marks if the subsequent working contains no further errors.

Follow through procedures may be applied repeatedly throughout the same problem.

The errors made by a candidate may be: arithmetical errors; errors in algebraic manipulation; errors in geometrical representation; use of an incorrect formula; errors in conceptual understanding.

The following illustrates a use of the **follow through** procedure:

Markscheme		Candidate's Script	Marking	
\$ 600 × 1.02	**M1**	Amount earned = \$ 600 × 1.02	✓	**M1**
= \$ 612	**A1**	= \$602	×	**A0**
\$ (306 × 1.02) + (306 × 1.04)	**M1**	Amount = 301 × 1.02 + 301 × 1.04	✓	**M1**
= \$ 630.36	**A1**	= \$ 620.06	✓	**A1(ft)**

Note that the candidate made an arithmetical error at line 2; the candidate used a correct method at lines 3, 4; the candidate's working at lines 3, 4 is correct.

However, if a question is transformed by an error into a **different, much simpler question** then:

(i) **fewer** marks should be awarded at the discretion of the Examiner;

(ii) marks awarded should be followed by '**(d)**' (to indicate that these marks have been awarded at the **discretion** of the Examiner);

(iii) a brief **note** should be written on the script explaining **how** these marks have been awarded.

4 Using the Markscheme

(a) This markscheme presents a particular way in which each question may be worked and how it should be marked. **Alternative methods** have not always been included. Thus, the working out must be carefully analysed in order that marks are awarded for a different method in a manner which is consistent with the markscheme.

In this case:

(i) a mark should be awarded followed by '**(d)**' (to indicate that these marks have been awarded at the **discretion** of the Examiner);

(ii) a brief **note** should be written on the script explaining **how** these marks have been awarded.

(b) Unless the question specifies otherwise, accept **equivalent forms**. For example: $\dfrac{\sin\theta}{\cos\theta}$ for $\tan\theta$.

(c) As this is an international examination, all **alternative forms of notation** should be accepted. For example: 1.7 , 1·7 , 1,7 ; different forms of vector notation such as $\vec{u}$, $\bar{u}$, $\underline{u}$; $\tan^{-1} x$ for $\arctan x$.

5 Accuracy of Answers

(a) In the case when the accuracy of answers is **specified in the question** (for example: "all answers should be given to four significant figures") **A** marks are awarded **only if** the correct answers are given to the accuracy required.

(b) When the accuracy is **not** specified in the question, then the general rule applies:

Unless otherwise stated in the question, all numerical answers must be given exactly or to three significant figures as appropriate.

In this case, the candidate is **penalised once only IN EACH QUESTION** for giving a correct answer to the wrong degree of accuracy. Hence, on the **first** occasion in a question when a correct answer is given to the wrong degree of accuracy **A** marks are **not** awarded. But on **all subsequent occasions** in the same question when correct answers are given to the wrong degree of accuracy then **A** marks **are** awarded.

NOVEMBER 1999

Additional instructions for Assistant Examiners

1. SAMPLES

All examiners are reminded that samples should be sent to the Team Leader by the fastest means possible. IBCA will reimburse examiners for any costs incurred.

2. PAPER 2 EXAMINERS – PART MARKS

Assistant examiners are asked to indicate on candidates' scripts the part marks that they have allocated for each part. To help identify these marks, they have now been incorporated into the markschemes. Where the markscheme has part marks *e.g.* ***[3 marks]*** after a part solution, assistant examiners should note the candidates' marks for that part of the question alongside the solution. They should also write down the total for the question at the end of each question.

1. (i) (a) $(1+\text{i})^2 = 1 + 2\text{i} + \text{i}^2$ *(M1)*

$= 2\text{i}$ *(A1)*

[2 marks]

(b) $(1+\text{i})^{4n}$

Let $\text{P}(n)$ be the proposition: $(1+\text{i})^{4n} = (-4)^n$

We must first show that $\text{P}(1)$ is true.

$(1+\text{i})^4 = \left((1+\text{i})^2\right)^2 = (2\text{i})^2$ *(M1)*

$= 4(\text{i})^2 = (-4)^1$ *(A1)*

Next, assume that for some $k \in \mathbb{N}^+$
$\text{P}(k)$ is true, then show that $\text{P}(k+1)$ is true.
$\text{P}(k): (1+\text{i})^{4k} = (-4)^k$ *(C1)*
Now, $(1+\text{i})^{4(k+1)} = (1+\text{i})^{4k}\,(1+\text{i})^4$

$= (-4)^k\,(-4)$ *(M1)*

$= (-4)^{k+1}$ *(A1)*

Therefore, by mathematical induction $\text{P}(n)$ is true for all $n \in \mathbb{N}^+$ *(C1)*

[6 marks]

(c) $(1+\text{i})^{32} = (1+\text{i})^{4(8)} = (-4)^8$ *(M1)*

$= 65536$ *(A1)*

[2 marks]

(ii) (a) $z_1 = \dfrac{\sqrt{6} - \text{i}\sqrt{2}}{2}$ $\quad |z_1| = \sqrt{\dfrac{6}{4} + \dfrac{2}{4}}$ *(M1)*

$= \sqrt{2}$ *(A1)*

$\arg z_1 = \arctan\left(\dfrac{-1}{\sqrt{3}}\right) = -\dfrac{\pi}{6}$ *(A1)*

Therefore, $z_1 = \sqrt{2}\left(\cos\left(-\dfrac{\pi}{6}\right) + \text{i}\sin\left(-\dfrac{\pi}{6}\right)\right)$ *(C1)*

$z_2 = 1 - \text{i}$ $\quad |z_2| = \sqrt{1+1} = \sqrt{2}$

$\arg z_2 = \arctan(-1) = -\dfrac{\pi}{4}$ *(A1)*

$z_2 = \sqrt{2}\left(\cos\left(-\dfrac{\pi}{4}\right) + \text{i}\sin\left(-\dfrac{\pi}{4}\right)\right)$ *(C1)*

[6 marks]

(b) $$\frac{z_1}{z_2} = \frac{\sqrt{2}\left(\cos\left(-\frac{\pi}{6}\right) + \text{i}\sin\left(-\frac{\pi}{6}\right)\right)}{\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + \text{i}\sin\left(-\frac{\pi}{4}\right)\right)}$$

$= 1\left(\cos\left(-\dfrac{\pi}{6} + \dfrac{\pi}{4}\right) + \text{i}\sin\left(-\dfrac{\pi}{6} + \dfrac{\pi}{4}\right)\right)$ *(M2)*

$= \cos\dfrac{\pi}{12} + \text{i}\sin\dfrac{\pi}{12}$ *(AG)*

[2 marks]

continued…

Question 1 (ii) continued

(c) $\frac{z_1}{z_2} = \left(\frac{\sqrt{6} - \mathrm{i}\sqrt{2}}{2}\right)\left(\frac{1}{1-\mathrm{i}}\right)\left(\frac{1+\mathrm{i}}{1+\mathrm{i}}\right)$ ***(M1)***

$$= \frac{\sqrt{6} + \sqrt{2} + \mathrm{i}\left(\sqrt{6} - \sqrt{2}\right)}{4}.$$ ***(A1)***

Therefore, $a = \cos\frac{\pi}{12} = \frac{\sqrt{6} + \sqrt{2}}{4}$ ***(A1)***

$b = \sin\frac{\pi}{12} = \frac{\sqrt{6} - \sqrt{2}}{4}$ ***(A1)***

[4 marks]

Note: Some students may use the half-angle formulas. Answers will only differ in form.
$\cos\frac{\pi}{12} = \frac{\sqrt{2+\sqrt{3}}}{2}$ $\sin\frac{\pi}{12} = \frac{\sqrt{2-\sqrt{3}}}{2}$

Total [22 marks]

2. (a) $\vec{AB} = -\vec{i} - 3\vec{j} + \vec{k}$, $\vec{BC} = \vec{i} + \vec{j}$ *(A2)*

[2 marks]

(b) $\vec{AB} \times \vec{BC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & -3 & 1 \\ 1 & 1 & 0 \end{vmatrix}$ *(M1)*

$= -\vec{i} + \vec{j} + 2\vec{k}$ *(A2)*

[3 marks]

(c) Area of $\Delta ABC = \frac{1}{2}\left| -\mathrm{i} + \vec{j} + 2\vec{k} \right|$ *(M1)*

$= \frac{1}{2}\sqrt{1+1+4}$

$= \frac{\sqrt{6}}{2}$ *(A1)*

[2 marks]

(d) A normal to the plane is given by $\vec{n} = \vec{AB} \times \vec{BC} = -\vec{i} + \vec{j} + 2\vec{k}$ *(M1)*

Therefore, the equation of the plane is of the form $-x + y + 2z = g$, and since the plane contains A, then $-1 + 2 + 2 = g \Rightarrow g = 3$. *(M1)*

Hence, an equation of the plane is $-x + y + 2z = 3$. *(A1)*

[3 marks]

(e) Vector $\vec{n}$ above is parallel to the required line.

Therefore, $x = 2 - t$ *(M1)*

$y = -1 + t$

$z = -6 + 2t$ *(A1)*

[2 marks]

(f) Distance of a point (x_0, y_0, z_0) from a plane $ax + by + cz + d = 0$ is given by

$\frac{\left| ax_0 + by_0 + cz_0 + d \right|}{\sqrt{a^2 + b^2 + c^2}}$ *(M1)*

since $-x + y + 2z - 3 = 0$ and D is $(2, -1, -6)$ *(M1)*

then; distance $= \frac{|-2-1-12-3|}{\sqrt{1+1+4}} = \frac{18}{\sqrt{6}}$

$= 3\sqrt{6}$ *(A1)*

[3 marks]

(g) Unit vector in the direction of $\vec{n}$ is $\vec{e} = \frac{1}{|\vec{n}|} \times \vec{n}$ *(M1)*

$= \frac{1}{\sqrt{6}}\left(-\vec{i} + \vec{j} + 2\vec{k}\right)$ *(A1)*

($-\vec{e}$ is also acceptable)

[2 marks]

(h) Let H be the intersection of DE with the plane, then

$-2 + t + (-1 + t) + 2(-6 + 2t) = 3$ *(M1)*

$\Rightarrow \quad 6t = 18$

$t = 3$ *(A1)*

$\Rightarrow \mathrm{H}(-1, 2, 0)$

but H is the mid point of DE *(M1)*

$\Rightarrow \mathrm{E}(-4, 5, 6)$ *(A1)*

[4 marks]

Total [21 marks]

3. (a) $f_k(x) = x\ln x - kx$

$\Rightarrow f_k'(x) = \ln x + 1 - k$ *(M1)(A1)*

[2 marks]

(b) $f_0'(x) = \ln x + 1$

$f_0(x)$ increases where $f_0'(x) > 0$ *(M1)*

$\Rightarrow \ln x > -1 \Rightarrow x > \dfrac{1}{e}$ *(A1)*

[2 marks]

(c) (i) Stationary points happen where $f_k'(x) = 0$

$\Rightarrow \ln x = k - 1$ *(M1)*

$\Rightarrow x = e^{k-1}$ *(A1)*

(ii) x intercepts are where $f_k(x) = 0$

$\Rightarrow x\ln x - kx = 0$

$\Rightarrow x(\ln x - k) = 0$ *(M1)*

$\Rightarrow x = 0$ or $\ln x = k$

$\Rightarrow x = e^k$

$\Rightarrow (e^k, 0)$ *(A1)*

[4 marks]

(d) Area $= \int_0^{e^k} \left|(x\ln x - kx)\right| dx \left(= \int_0^{e^k}(kx - x\ln x)\,dx \right)$ *(M1)*

Integrate $x\ln x$ by parts.

$\int x\ln x\,dx = \dfrac{x^2}{2}\ln x - \int \dfrac{x}{2}\,dx$ *(M1)*

$= \dfrac{x^2}{2}\ln x - \dfrac{x^2}{4} + C$ *(A1)*

$\Rightarrow$ Area $= \left| \left[\dfrac{x^2}{2}\ln x - \dfrac{x^2}{4} - \dfrac{kx^2}{2} \right]_0^{e^k} \right|$ *(M1)*

$= \dfrac{e^{2k}}{4}$ *(A1)*

Note: Given $x\ln x - kx = f_k(x) \approx 0$ when $x = 0$.

[5 marks]

(e) Gradient of the tangent at $A(e^k, 0), m$ is $f_k'(e^k) = \ln e^k + 1 - k$ *(M1)*

$= 1$

Therefore, an equation of the tangent is $y = x - e^k$. *(A1)*

[2 marks]

continued...

Question 3 continued

(f) The tangent forms a right angle triangle with the coordinate axes. The prependicular sides are each of length e^k. ***(M1)***

Area of the triangle $= \frac{1}{2} \times e^k \times e^k = \frac{1}{2} e^{2k}$ ***(A1)***

$\frac{1}{2} e^{2k} = 2\left(\frac{1}{4} e^{2k}\right)$ *i.e.* The area of the triangle is twice the area enclosed by the curve and the x-axis. ***(AG)***

[2 marks]

(g) Since the x-intercepts are of the form $x_k = e^k$, for $k \in \mathbb{N}$ ***(M1)***

then $x_{k+1} = e^{k+1}$

and $\frac{x_{k+1}}{x_k} = e$ ***(A1)***

Therefore, the x-intercepts $x_0, x_1, \ldots x_k, \ldots$ form a geometric sequence with $x_0 = 1$ and a common ratio of e. ***(R1)***

[3 marks]

Total [20 marks]

4. (a) For $f(x)$ to be a probability distribution function, $\int_0^1 f(x)\,dx = 1$.

$\Rightarrow \int_0^1 (e - ke^{kx})\,dx = 1$ *(M1)*

$\Rightarrow \left[ex - e^{kx}\right]_0^1 = 1$ *(M1)*

$\Rightarrow e - e^k + 1 = 1$ *(A1)*

$\Rightarrow e = e^k \Rightarrow \quad k = 1$ *(AG)*

Thus $f(x) = e - e^x,\ 0 \le x \le 1$

[3 marks]

(b) $\int_{1/4}^{1/2} \left(e - e^x\right)dx = \left[ex - e^x\right]_{1/4}^{1/2} \quad = \frac{e}{2} - \sqrt{e} - \frac{e}{4} + \sqrt[4]{e}$ *(M1)*

$= \frac{e}{4} - \sqrt{e} + \sqrt[4]{e}$ *(A1)*

[2 marks]

(c) $\mu = \int_0^1 x\left(e - e^x\right)dx \quad = \int_0^1 \left(ex - xe^x\right)dx$ *(M1)*

$= \left[\frac{ex^2}{2}\right]_0^1 - \int_0^1 xe^x\,dx = \frac{e}{2} - \left[xe^x - e^x\right]_0^1$ *(M1)*

$= \frac{e}{2} - 1$ *(A1)*

Variance $= \int_0^1 x^2\left(e - e^x\right)dx - \left(\frac{e}{2} - 1\right)^2$ *(M1)*

$= \left[\frac{ex^3}{3} - e^x\left(x^2 - 2x + 2\right)\right]_0^1 - \left(\frac{e}{2} - 1\right)^2$ *(M1)*

$= 2 - \frac{2}{3}e - \frac{e^2}{4} + e - 1$

$= 1 + \frac{e}{3} - \frac{e^2}{4}$ *(A1)*

[6 marks]

(d) p(battery lasts more than 6 months) $= p\left(x > \frac{1}{2}\right)$

$= \int_{1/2}^1 \left(e - e^x\right)dx$ *(M1)*

$= \left[ex - e^x\right]_{1/2}^1$

$= \sqrt{e} - \frac{e}{2}$ or 0.290 (3 s.f.) *(A1)*

[2 marks]

(e) p(no battery failed) $= p$(all lasted more than 6 months) *(M1)*

$= \left(\sqrt{e} - \frac{e}{2}\right)^3$ or 0.0243 (3 s.f.) *(A1)*

[2 marks]

(f) p(exactly one battery failed) $= \binom{3}{2}\left(1 - \sqrt{e} + \frac{e}{2}\right)\left(\sqrt{e} - \frac{e}{2}\right)^2$ *(M1)*

≈ 0.179 (3 s.f.) *(A1)*

[2 marks]

Total [17 marks]

5. (i) (a) $(3*9)*13 = 13*13 = 1$ *(M1)*

and $3*(9*13) = 3*5 = 1$ *(M1)*

hence $(3*9)*13 = 3*(9*13)$ *(AG)*

[2 marks]

(b) To show that $(U, *)$ is a group we need to show that:

(1) U is closed under $*$.

A table is an easy way of showing closure for this finite set.

*	1	3	5	9	11	13
1	1	3	5	9	11	13
3	3	9	1	13	5	11
5	5	1	11	3	13	9
9	9	13	3	11	1	5
11	11	5	13	1	9	3
13	13	11	9	5	3	1

(C4)

Note: Award *(C4)* for a completely accurate table, *(C3)* for 1 or 2 errors, *(C2)* for 3 or 4 errors, *(C1)* for 5 or 6 errors, *(C0)* for 7 or more errors.

since for each $a, b \in U$, $a * b \in U$, **closure** is shown. *(C1)*

(2) Since multiplication is **associative**, it is true in this case too. *(C2)*

(3) Since $1*a = a*1 = a$ for all $a \in U$, 1 is the **identity**. *(C2)*

(4) 1 appears in each row of the table once, so every element has a unique **inverse**.

$(1^{-1} = 1, 3^{-1} = 5, 5^{-1} = 3, 9^{-1} = 11, 11^{-1} = 9, 13^{-1} = 13)$ *(C2)*

[11 marks]

(c) (i) If G is a group and if there exists $a \in G$, such that

$$G = \{a^n : n \in \mathbb{Z}\}$$

Then G is a cyclic group and a is called a generator. *(C2)*

[2 marks]

(ii) By inspection:

3 is a generator since:

$3^2 = 9, 3^3 = 13, 3^4 = 11$ *(M1)*

$3^5 = 5, 3^6 = 1$ *(A1)*

Also, 5 is a generator:

$5^2 = 11, 5^3 = 13, 5^4 = 9$ *(M1)*

$5^5 = 3, 5^6 = 1$ *(A1)*

9 cannot be a generator since

$9^3 = 1$ *(C1)*

similarly $11^3 = 1$ *(C1)*

and $13^2 = 1$. *(C1)*

[7 marks]

continued...

Question 5 (i) continued

(d) Since the order of this group is 6, by Lagrange's Theorem, the proper subgroups can only have orders 2 or 3. ***(R1)***

Since 13 is the only self inverse $13^2 = 1$, ***(R1)***
the only subgroup of order 2 is $\{1, 13\}$ ***(A1)***
No sub-group may include 3 or 5 since these are the generators of the group.
The only elements left are 9 and 11. ***(R1)***
Now, $9 * 11 = 1, 9^2 = 11$, and $11^2 = 9$. ***(M2)***
Therefore, $\{1, 9, 11\}$ is the other sub-group. ***(A1)***

[7 marks]

(ii) (a) Since $\forall a \in G, e \circ a = a \circ e$ because e is the identity element of the group. ***(R2)***
Then $e \in H$. ***(AG)***

[2 marks]

(b) Let $x, y \in H$, then $(x \circ y) \circ a = x \circ (y \circ a)$ (by associativity) ***(R1)***
$= x \circ (a \circ y)$ (since $y \in H$) ***(R1)***
$= (x \circ a) \circ y$ (associativity) ***(R1)***
$= (a \circ x) \circ y$ ($x \in H$) ***(R1)***
$= a \circ (x \circ y)$ (associativity) ***(R1)***
Therefore, $(x \circ y) \circ a = a \circ (x \circ y)$
$\Rightarrow (x \circ y) \in H$. ***(AG)***

[5 marks]

(c)

	$e \circ a = a \circ e$	identity	***(R1)***
$\Rightarrow$	$(x^{-1} \circ x) \circ a = a \circ (x^{-1} \circ x)$		***(R1)***
$\Rightarrow$	$x^{-1} \circ (x \circ a) = (a \circ x^{-1}) \circ x$	associativity	
$\Rightarrow$	$x^{-1} \circ (a \circ x) = (a \circ x^{-1}) \circ x$	$x \in H$	***(R1)***
$\Rightarrow$	$(x^{-1} \circ a) \circ x = (a \circ x^{-1}) \circ x$	associativity	***(R1)***
Therefore,	$x^{-1} \circ a = a \circ x^{-1}$	cancellation law	
	and $x^{-1} \in H$		***(AG)***

[4 marks]

Total [40 marks]

6. (i) (a)

	A	B	C	D	E	F
A	0	1	2	2	2	1
B	1	0	1	2	3	2
C	2	1	0	1	2	1
D	2	2	1	0	2	1
E	2	3	2	2	0	1
F	1	2	1	1	1	0

(A4)

Note: Award *(A4)* for a completely correct table, *(A3)* for 1 error, *(A2)* for 2 errors, *(A1)* for 3 errors, *(A0)* for 4 or more errors.

[4 marks]

(b) (i) The index for each city is given below:

A: $\frac{8}{2} = 4$; B: $\frac{9}{2} = 4.5$

C: $\frac{7}{3} = 2.\overline{3}$; D: $\frac{8}{2} = 4$

E: $\frac{10}{1} = 10$; F: $\frac{6}{4} = 1.5$ *(C2)*

Note: Award *(C2)* for all correct, *(C1)* for 1 error, *(C0)* for 2 or more errors.

City F is the most accessible since its index is 1.5. *(C1)*

City E is the least accessible since its index is 10. *(C1)*

(ii) The indices are now:

A: $2.\overline{3}$ B: 4.5 C: 1.5

D: 4 E: 10 F: 1.5 *(M1)*

C and F are the most accessible cities. *(C1)*
E is still the least accessible. *(C1)*

[7 marks]

Question 6 continued

(ii) (a)

$$\begin{array}{c|c} \text{U} & \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{matrix} & \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

(A3)
[3 marks]

> **Note:** Award *(A3)* for a completely correct table, *(A2)* for 1 or 2 errors, *(A1)* for 3 or 4 errors, *(A0)* for 5 or more errors.

(b) Adjacency matrix for V.

$$\begin{array}{c|c} & \begin{matrix} A & E & B & F & C & G & D & H \end{matrix} \\ \begin{matrix} A \\ E \\ B \\ F \\ C \\ G \\ D \\ H \end{matrix} & \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

(M2)(A2
)

> **Note:** Award *(M2)* for correct correspondence,
> *(M1)* for 1 or 2 errors, *(M0)* for 3 or more errors.
> Award *(A2)* for correct matrix, *(A1)* for 1 or 2 errors, *(A0)* for 3 or more errors.

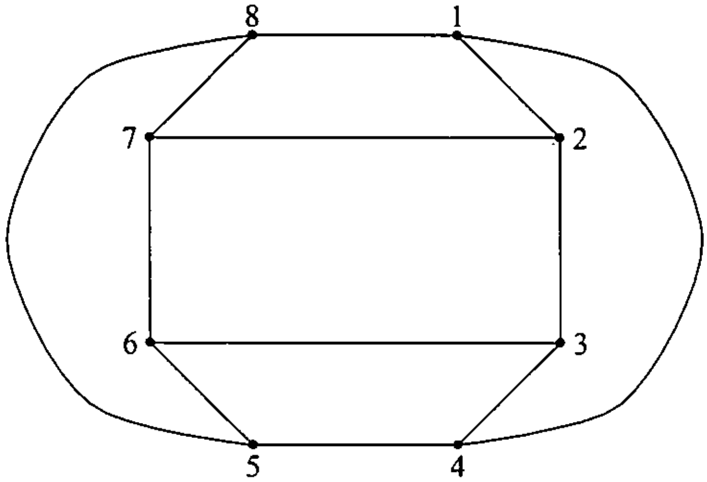
Since there is a one-to-one correspondence in which 2 vertices are adjacent if and only if their images are adjacent, and since the matrix structures are identical, then the two graphs are isomorphic. *(R1)* *(R1)*

[6 marks]

continued…

Question 6 (ii) continued

(c) By redrawing U as shown below we can see that it is planar and since U ≅ V, V is also planar as isomorphism preserves planarity. ***(C1)(R1)*** ***(A1)(C1)***



U

[4 marks]

(iii) (a)

Vertices added to the Tree	Edge added	Weight
3	∅	0
5	3, 5	10
6	3, 6	20
7	5, 7	30
10	6, 10	30
1	3, 1	40
2	1, 2	30
11	2, 11	30
9	1, 9	40
4	6, 4	40
8	7, 8	40
		310

(R2)(A4)(M1)

(A1)

Note: Award ***(R2)*** for correct algorithms, ***(R1)*** for 1 error, ***(R0)*** for 2 or more errors.
Award ***(A4)*** for correct calculations, ***(A3)*** for 1 error, ***(A2)*** for 2 errors, ***(A1)*** for 3 errors, ***(A0)*** for 4 or more errors.
Award ***(M1)*** for tree/table/method.
Award ***(A1)*** for minimum weight.

[8 marks]

continued...

Question 6 (iii) continued

(b) Vertices and potentials

Step	Starting Vertex	1	2	3	4	5	6	7	8	9	10	11
1	1		$\boxed{6}$	$\boxed{3}$	8							
2	2				13	20	22					
3	3				$\boxed{5}$	15	22					
4	4					$\boxed{11}$		$\boxed{29}$				
5	5						$\boxed{18}$	31	34	67		
6	6							47	$\boxed{33}$	$\boxed{48}$		
7	7									70	71	
8	8									83	$\boxed{58}$	
9	9										88	$\boxed{80}$
10	10											$\boxed{80}$

(R2)(A4)

Minimum = 80 ***(A1)***

Any of two paths: 1 - 3 - 4 - 5 - 6 - 8 - 10 - 11 or 1 - 3 - 4 - 5 - 6 - 9 - 11 ***(A1)***

Note: Award ***(R2)*** for **correct algorithms**, ***(R1)*** for 1 error, such as an unclear statement or application, ***(R0)*** for 2 or more errors.
Award ***(A4)*** for correct calculations, ***(A3)*** for 1 error, ***(A2)*** for 2 errors, ***(A1)*** for 3 errors, ***(A0)*** for 4 or more errors.

[8 marks]

Total [40 marks]

7. (i) (a) $p(X > 327)$

$= p\left(Z > \dfrac{327-330}{6}\right)$ *(M1)*

$= p\left(Z > -\dfrac{1}{2}\right) = 0.6915$

$= 0.692$ (3 s.f.) *(A1)*

[2 marks]

(b) Let x_0 be the required amount of coffee.

$p(X > x_0) = 0.95$ *(M1)*

$\Rightarrow \dfrac{x_0 - 330}{6} = -1.645$ *(M1)*

$\Rightarrow x_0 = 330 - 1.645 \times 6 = 320.13$

$= 320$ (3 s.f.) *(A1)*

[3 marks]

(c) $p(\bar{X} \geq 332) = p\left(Z \geq \dfrac{332-330}{\frac{6}{\sqrt{60}}}\right)$ *(M1)*

$= p(Z \geq 2.58)$ *(M1)*

$= 1 - 0.9951$ or $1 - 0.99509$ (from GDC)

$= 0.0049$ or 0.00491 (3 s.f.) *(A1)*

[3 marks]

(d) Since a sample with this mean has a chance of 0.5 % of being produced by this *(R1)* machine, we may agree with the vendor that the true mean of the volume of coffee *(R1)* seems to be higher than 330 ml. *(R1)*

[3 marks]

(e) A 95 % confidence interval is:

$332.78 \pm 1.96 \times \dfrac{6}{\sqrt{25}}$ *(M2)*

$= (330.43, 335.13)$ *(A2)*

[4 marks]

(f) Since $330 \notin (330.43, 335.13)$ we can conclude that there is enough evidence to *(M1)* support the hypothesis. *(R1)*

[2 marks]

(g) Among the 25 cups sampled, there are 18 that have more than 330 ml. *(M1)*

Let p be the proportion of cups that have more coffee than the vector desires.

An estimate for p is $\dfrac{18}{25} = 0.72$. *(A1)*

A 95 % confidence interval for p is

$0.72 \pm 1.96\sqrt{\dfrac{0.72 \times 0.28}{25}}$ *(M1)*

$= (0.544, 0.896)$ *(A1)*

[4 marks]

continued...

Question 7 continued

(ii) (a) To calculate expected frequencies, we multiply 4000 by the probability of each cell:

$$p(80.5 \le X \le 90.5) = p\left(\frac{80.5-100}{10} \le Z \le \frac{90.5-100}{10}\right) \quad \textit{(M1)}$$
$$= p(-1.95 \le Z \le -0.95)$$
$$= 0.1711 - 0.0256$$
$$= 0.1455$$

Therefore, the expected frequency $= 4000 \times 0.1455$ *(M1)*
≈ 582 *(A1)*

Similarly:

$$p(90.5 \le X \le 100.5) = 0.5199 - 0.1711$$
$$= 0.3488$$

Frequency $= 4000 \times 0.3488$
≈ 1396 *(A1)*

And

$$p(100.5 \le X \le 110.5) = 0.8531 - 0.5199$$
$$= 0.3332$$

Frequency $= 4000 \times 0.3332$
≈ 1333 *(A1)*

[5 marks]

(b) To test the goodness of fit of the normal distribution, we use the χ^2 distribution. Since the last cell has an expected frequency less than 5, it is combined with the cell preceding it. There are therefore $7 - 1 = 6$ degrees of freedom. *(C1)*

$$\chi^2 = \frac{(20-6)^2}{6} + \frac{(90-96)^2}{96} + \frac{(575-582)^2}{582} + \frac{(1282-1396)^2}{1396}$$

$$+ \frac{(1450-1333)^2}{1333} + \frac{(499-507)^2}{507} + \frac{(84-80)^2}{80} \quad \textit{(M1)}$$

$$= 53.03 \quad \textit{(A1)}$$

H_0: Distribution is Normal with $\mu = 100$ and $\sigma = 10$.
H_1: Distribution is not Normal with $\mu = 100$ and $\sigma = 10$. *(M1)*

$\chi^2_{(0.95,6)} = 14.07$
Since $\chi^2 = 53.0 > \chi^2_{\text{critical}} = 14.07$, we reject H_0. *(A1)*

Therefore, we have enough evidence to suggest that the normal distribution with mean 100 and standard deviation 10 does not fit the data well. *(R1)*

[6 marks]

Note:	If a candidate has not combined the last 2 cells, award ***(C0)(M1)(A0)(M1)(A1)(R1)*** (or as appropriate).

Question 7 continued

(iii) This is the case for a contingency table. The observed frequencies are given. The expected frequencies can be calculated by multiplying the column and row totals for each cell and dividing the product by the total number of observations,

e.g. First cell: $\frac{40 \times 80}{100} = 32$. (Others are calculated in a similar manner.) ***(M1)***

Expected frequencies	No flu immunisation injections	Flu immunisation injections
Cold	32	8
No cold	48	12

(A1)

Since there is only $(2-1)(2-1)=1$ degree of freedom we use Yates' continuity correction factor: ***(R1)***

$$\chi^2 = \frac{(|35-32|-0.5)^2}{32} + \frac{(|5-8|-0.5)^2}{8} + \frac{(|45-48|-0.5)^2}{48} + \frac{(|15-12|-0.5)^2}{12}$$ ***(M1)***

$$= 1.628$$ ***(A1)***

H_0: Flu immunisation injections and suffering from colds are independent.
H_1: There is evidence of dependence between flu immunisation injections and colds. ***(M1)***

$\chi^2_{(0.95,1)} = 3.84$.

Since $\chi^2_{\text{test}} = 1.628 < \chi^2_{\text{critical}} = 3.84$. ***(A1)***

We do not have enough evidence to reject H_0. Hence we do not have enough evidence to support the claim that flu injections help reduce the number of people suffering from colds. ***(R1)***

[8 marks]

Total [40 marks]

8. (i) (a) $f(1) = -3$ *(A1)*

$f(2) = 2$ *(A1)*

[2 marks]

(b) $f'(x) = 6x^2 - 30x + 36$ *(C1)*

[1 mark]

(c) Since $f(1) = -3$, and $f(2) = 2$, by the intermediate value theorem, $f(x)$ has at least one zero between 1 and 2. *(M1)* *(R1)*

Since $f'(x) > 0$ in the interval $1 < x < 2$, $f(x)$ can intersect x-axis only once. *(R1)*

[3 marks]

(d) $$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$ *(M1)*

$$x_1 = 1.5 - \frac{f(1.5)}{f'(1.5)} = 1.2778$$ *(M1)*

$x_2 = 1.3204$ *(A1)*

$x_3 = 1.3223$

$x_4 = 1.3223 \Rightarrow$

$x \approx 1.322$ (3 d.p.) *(A1)*

[4 marks]

(e) $2x^3 = 15x^2 - 36x + 26$

$$x = \sqrt[3]{\frac{1}{2}\left(15x^2 - 36x + 26\right)}$$ *(M1)*

$$x_{n+1} = \sqrt[3]{\frac{1}{2}\left(15x_n^2 - 36x_n + 26\right)}$$ *(C1)*

$x_1 = \sqrt[3]{2.875} = 1.4219$ *(A1)*

$x_2 = 1.3697$

$x_3 = 1.34$ (3 s.f.) *(A1)*

[4 marks]

(ii) (a) $$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{(n+1)^2\, 2^{n+2}}{3^{n+1}} \times \frac{3^n}{n^2\, 2^{n+1}}$$ *(M1)*

$$= \lim_{n\to\infty} \frac{2(n+1)^2}{3n^2}$$ *(M1)*

$$= \frac{2}{3} < 1$$ *(A1)*

This series converges by the ratio test since $\lim \frac{a_{n+1}}{a_n} < 1$. *(R1)*

[4 marks]

continued...

Question 8 (ii) continued

(b) Since $f(x)=\dfrac{x^{k-1}}{x^k+k}$ is decreasing and continuous for $x>2$, the integral test may be applied. ***(R1)***

$$\int_1^{\infty}\frac{x^{k-1}}{x^k+k}\mathrm{d}x=\lim_{a\to\infty}\left[\frac{1}{k}\ln\left(x^k+k\right)\right]_1^a \qquad \textit{(M2)}$$

$$=\lim_{a\to\infty}\left[\frac{1}{k}\ln\left(a^k+k\right)-\frac{1}{k}\ln(k+1)\right] \qquad \textit{(A1)}$$

$$=\infty$$

Thus the integral diverges and consequently the series diverges too. ***(R1)***

[5 marks]

(iii) (a) $f'(x)=\dfrac{-x}{\sqrt{2-x^2}}$ ***(A1)***

$$f''(x)=\frac{-\sqrt{2-x^2}+x\left(\frac{1}{2}\right)(-2x)\left(2-x^2\right)^{-1/2}}{2-x^2} \qquad \textit{(M1)}$$

$$=\frac{-2+x^2-x^2}{\left(2-x^2\right)\sqrt{2-x^2}} \qquad \textit{(A1)}$$

$$=\frac{2}{\left(x^2-2\right)\sqrt{2-x^2}} \qquad \textit{(AG)}$$

[3 marks]

(b) $f(x)\approx f(0)+f'(0)x+f''(0)\dfrac{x^2}{2!}+\ldots+\dfrac{f^{(n)}(0)}{n!}x^n+\ldots$ ***(M1)***

$f(0)=\sqrt{2}$

$f'(0)=0$

$f''(0)=-\dfrac{\sqrt{2}}{2}$

$f'''(0)=0$

$f^4(0)=-\dfrac{3\sqrt{2}}{4}$ ***(A2)***

$$\Rightarrow\sqrt{2-x^2}\approx\sqrt{2}-\left(\frac{\sqrt{2}}{2}\right)\frac{x^2}{2}-\frac{3\sqrt{2}}{4}\left(\frac{x^4}{24}\right) \qquad \textit{(M1)}$$

$$=\sqrt{2}-\frac{\sqrt{2}}{4}x^2-\frac{\sqrt{2}}{32}x^4=\sqrt{2}\left(1-\frac{x^2}{4}-\frac{1}{32}x^4\right)$$

$$\Rightarrow k=-\frac{1}{32} \qquad \textit{(A1)}$$

[5 marks]

continued...

Question 8 (iii) continued

(c) $\int_0^1 x^2\sqrt{2-x^2}\,dx \approx \int_0^1 \sqrt{2}\left(x^2 - \frac{x^4}{4} - \frac{x^6}{32}\right)dx$ ***(M1)***

$$= \sqrt{2}\left[\frac{x^3}{3} - \frac{x^5}{20} - \frac{x^7}{224}\right]_0^1$$ ***(M2)***

$$= \sqrt{2}\left(\frac{1}{3} - \frac{1}{20} - \frac{1}{224}\right)$$ ***(A1)***

$$= 0.39438$$ ***(AG)***

[4 marks]

(d) $x = \sqrt{2}\sin\theta \Rightarrow \sin\theta = \frac{x}{\sqrt{2}}$

$$\Rightarrow x = 0, \theta = 0 \text{ or } x = 1, \theta = \frac{\pi}{4}$$

$dx = \sqrt{2}\cos\theta\, d\theta$

$\sqrt{2-x^2} = \sqrt{2}\cos\theta$ ***(M1)***

Therefore, $\int_0^1 x^2\sqrt{2-x^2}\,dx = \int_0^{\pi/4} 4\sin^2\theta\cos^2\theta\, d\theta$ ***(M1)***

$$= \int_0^{\pi/4} \sin^2 2\theta\, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/4}(1 - \cos 4\theta)\, d\theta$$ ***(M1)***

$$= \frac{1}{2}\left[\theta - \frac{\sin 4\theta}{4}\right]_0^{\pi/4}$$

$$= \frac{\pi}{8}$$ ***(A1)***

Therefore, $\frac{\pi}{8} \approx 0.39438 \Rightarrow \pi \approx 3.1550$ ***(A1)***

[5 marks]

Total [40 marks]